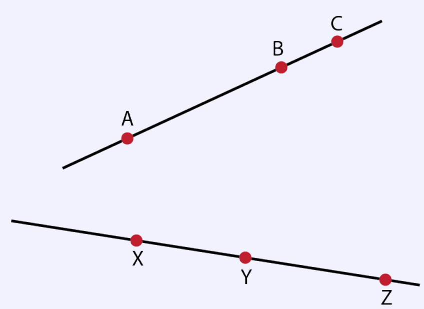
Grade 6 Math Circles
October 10/11/12/16
Exploring Conics

Playing With Lines

The job of a mathematician is to explore different objects and look for hidden patterns. Once we find something interesting, we must ask ourselves, "Why is this happening?" This is what we will be doing today.

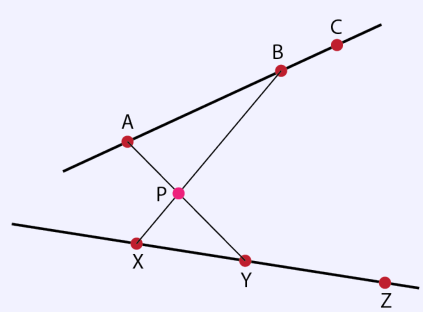
Exercise 1

(i) Begin by drawing two straight lines in any directions that you like. Anywhere on each line, draw three points, and from left to right label them A, B, and C, and X, Y, Z.



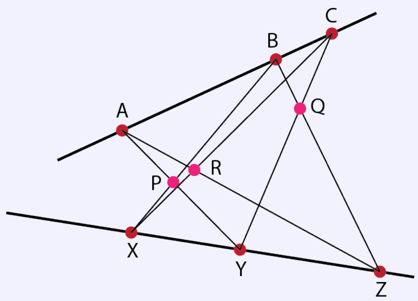
Tip: you will have an easier time if your points are spaced out.

(ii) Be careful in this step! Using your ruler, connect A and Y, and B and X. Label the point where the two lines meet P.



(iii) Similar to the last step, connect B and Z, and C and Y. Label the point where the two lines meet Q.

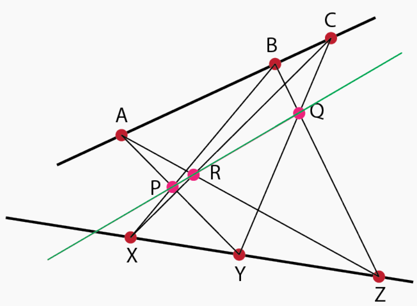
Finally, connect A and Z, and C and X. Label the point where the two lines meet R. You should have something that looks as follows:



What do you notice about the points P, Q, R? If you draw 2 different lines, or choose 6 different points, do you notice the same thing? Do your neighbours have the same result?

Solution

If everything went well in your drawing, we can (hopefully) see that P, Q, and R all lie on a single line.



What we've discovered is known as *Pappus' Theorem*. If we take any two lines and choose 3 points on each line, then the crossing intersections (the points where the pairs of lines meet) all lie on a single line.

Theorems in math are statements that we know are always true. We will discover many more theorems today.

Stop and Think

Is it special that three seemingly random points lie on a single line? If we were to choose three random points will they always lie on a single line?

Drawing Ellipses

The next shape that we will explore is an *ellipse*, similar looking to an oval.

Example 1

You will need: (1) two push pins, (2) a piece of paper, (3) a pencil, and (4) a piece of string.

(i) Tie two knots in your string, most types of knots should work fine.

(ii) Take your two pins and put them through the paper. Make sure the distance between the pins is shorter than the length of your string. It's easiest to have the pin sticking up, but be careful when doing this!

(iii) Slide each knot over the pins (one knot per pin).

(iv) Put your pencil against the string and make sure the string is tight. Trace along the string while keeping it tight. Once you've gone all the way around, you've created an oval!

If you would like a visual guide, here's a helpful YouTube video: https://www.youtube.com/watch?v=Et3OdzEGX_w

Try to draw a few ovals until you feel comfortable making them on your own.

Exercise 2

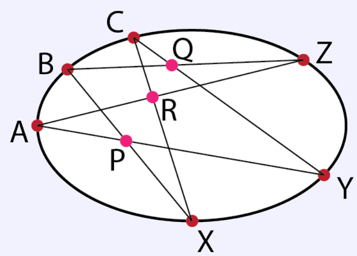
Split your oval into two halves. Perform the first exercise again, treating each half of the oval as the lines (the halves don't have to be even). I.e.,

(i) Draw 3 points on each half of the oval and from left to right label them A, B, and C, and X, Y, Z.

(ii) Using your ruler, connect A and Y, and B and X. Label the point where the two lines meet P.

Connect B and Z, and C and Y. Label the point where the two lines meet Q.

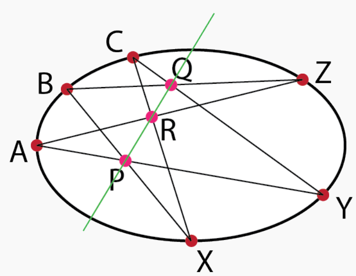
Finally, connect A and Z, and C and X. Label the point where the two lines meet R. You should have something similar to below:



What do you notice about the points P, Q, R? If you draw a different ellipse, or choose 6 different points, do you notice the same thing? Do your neighbours have the same result?

Solution

Just as with the two lines, if everything went well in your drawing, we should be able to see that P, Q, and R all lie on a single line.



We will call this property the *Six Point Property.*

As we mentioned, the job of a mathematician is to look for patterns. Do you notice any patterns so far today? If we feel like we have a pattern, we make a *conjecture*. A conjecture is a statement that we think is true.

Exercise 3

Without actually drawing a circle, make a conjecture about what you think might happen if you were to once again repeat the previous exercise, but on a circle instead of an oval. That is, make a conjecture about whether or not the Six Point Property will hold for circles. Provide justification as to why you think your conjecture is true.

Note: "Because it worked in the last two examples" is not good justification!

Test your conjecture. That is, draw a circle and perform the previous exercise.

Solution

A reasonable conjecture might be that we will again see the same result. In fact, circles are just one type of oval, just like squares are one type of rectangle. Since circles are ovals, the result must be true!

So far we've seen that the Six Point Property has held for very different objects - both lines and ovals. This is quite surprising!

Stop and Think

Do you think that the Six Point Property will hold for any type of shape? Make a conjecture about picking 6 points on different shapes.

Just like our conjecture about circles, whenever we make a conjecture, we must test it. Test your conjecture with another shape, such as a rectangle. The result will either provide more evidence for your conjecture, or disprove it.

Caution! If you draw a rectangle and only put your 6 points on two of the sides, then this is just exercise 1 again. Use at least 3 sides!

Drawing Hyperbola

Unless you were very lucky, the shape you chose likely does not have the Six Point Property, but there may still be other shapes that have this property.

We will now learn how to draw another shape, called a *hyperbola*[1].

Example 2

You will need: (1) two pins, (2) a piece of paper, (3) a pencil, (4) a piece of string, (5) two pieces of tape, and (6) a ruler.

(i) Tie one knot in your string.

(ii) Take your two pins and put them through the paper. Make sure the distance between the pins is much shorter than the length of your string.

(iii) Poke a hole in a piece of tape. Attach the tape to one end of the ruler, and slide the hole over one of the pins.

(iv) Take your string, and slide the knot over the second pin. Tape the string to the end of the ruler without the pin.

(v) Push your pencil against the string and ruler. Slowly rotate the ruler, keeping the pencil against both the string and the ruler.

If you would like a visual guide, here's a helpful YouTube video: https://www.youtube.com/watch?v=fUXi_Tf4Kxw

After you've completed these steps, you should have something that looks like this:

Try drawing a few hyperbolas until you feel comfortable making them.

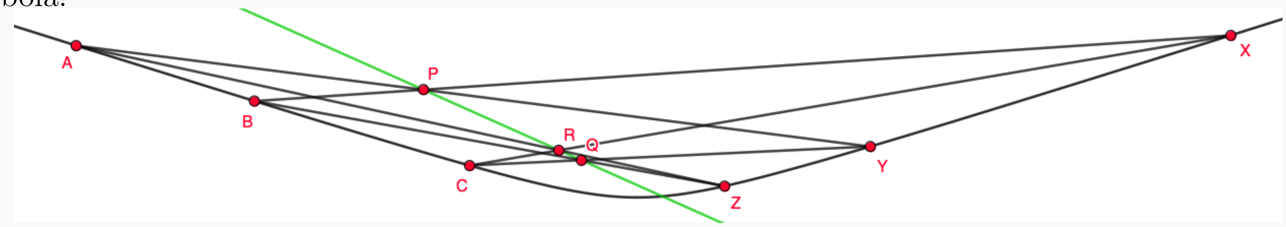
Exercise 4

See if the Six Point Property holds for the hyperbola.

[1]Technically, we will only be drawing half of the hyperbola, the second half is a mirror image of the first.

Solution

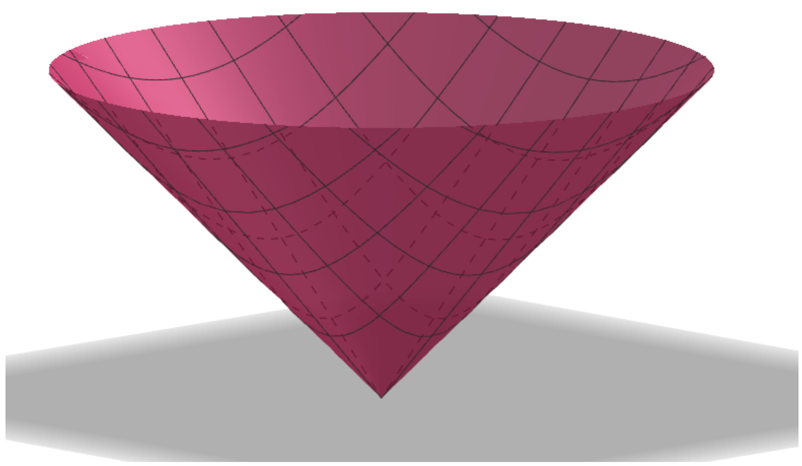
If everything went well, we should indeed see that the Six Point Property holds for your hyperbola!



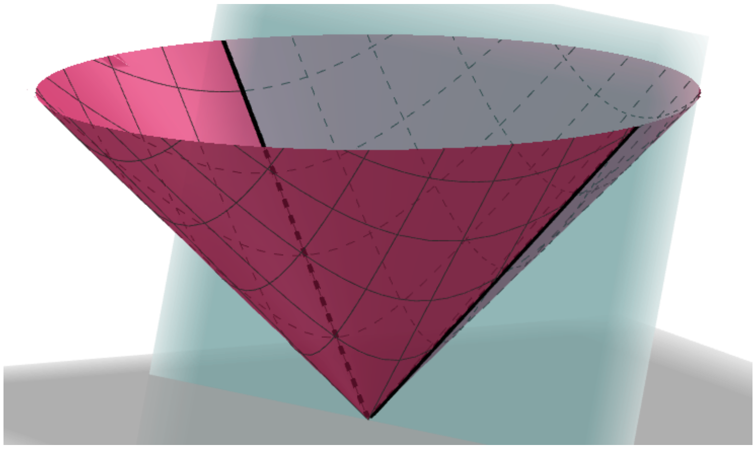
Conic Sections

What's going on here? So far, we have found three very different looking shapes that satisfy the Six Point Property, but what do they have in common?

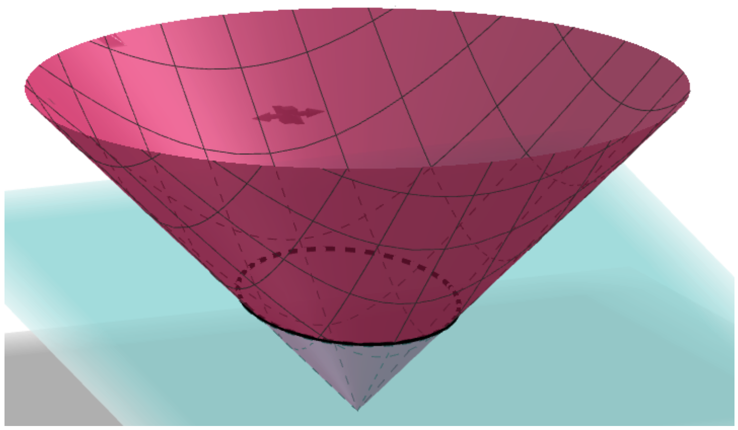
All of these are examples of what are known as a *conic section*. Let's look at a cone



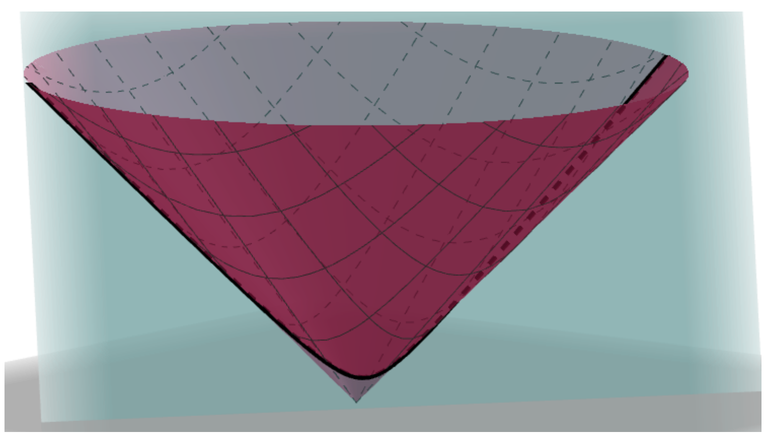
The cone is very special here. If we take different slices of it, then we get different shapes, namely all the shapes that satisfy the Six Point Property. Let's take a look. We'll represent slicing the cone using the flat, blue surface, and the resulting slice will be the black lines. You will explore slicing the cone more in the problem set.



Slicing with a plane like this gives us two lines like in the first exercise.



Slicing with a plane like this gives us an ellipse.



And finally, Slicing with a plane like this gives us a hyperbola.

It's worth noting that another conic section exists, but we'll explore this is the problem set.

The fact that any conic section satisfies the Six Point Property is known as *Pascal's Theorem*. Notice that Pappus' Theorem from the beginning was just a special case of Pascal's theorem. In fact, even Pascal's theorem is a special case of another, even more complicated theorem, which is well beyond the scope of this activity!